The matrix representation of the first cohomology of $\mathfrak{gl}_{0|2}$ with coefficients in the generalized Witt Lie superalgebra

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Abstract

This paper is primarily concerned with the first cohomology of $\mathfrak{gl}_{0|2}$ with coefficients in the generalized Witt Lie superalgebra, where $\mathfrak{gl}_{0|2}$ is a subalgebra of the general linear Lie superalgebra. The derivations and inner derivations from $\mathfrak{gl}_{0|2}$ into submodules of the generalized Witt Lie superalgebra are represented by matrices, respectively. Then the first cohomology of $\mathfrak{gl}_{0|2}$ with coefficients in the generalized Witt Lie superalgebra is completely determined by matrices.

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1. Introduction

Cohomology of Lie (super)algebras has many important applications in mathematics and physics (see [4]). It carries a great deal of fundamental (topological) information about algebras under consideration. For Lie algebras, many structural features of cohomology have been well investigated (see [2, 5, 18] for examples). The researches on cohomology of Lie

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superalgebras are set forth in [7]. As is well known, the theory of classical Lie superalgebras plays an important role in the research of Lie superalgebras (see [6, 10] for examples). Therefore, cohomology of classical Lie superalgebras is given considerable attention by mathematicians and physicists. In [3], cohomology groups of classical finite-dimensional Lie superalgebras, the list of classical superalgebras is an extended version of Kac's list [6] of simple classical Lie superalgebras, with trivial coefficients were calculated. The technique of computations was explained later by Fuks [4]. With an increasing amount of theories and applications concerning representations of Lie superalgebras, it is becoming possible to ascertain cohomology groups of classical Lie superalgebras with non-trivial coefficients. During the past decade, the theory of this problem has developed in a variety of directions and a large number of results have been obtained (see [1, 11, 12] for example). For classical Lie superalgebras over a field of prime characteristic, the recent papers [13, 14] computed low-dimensional cohomology groups of the special linear Lie superalgebra $\mathfrak{sl}_{m|n}$ and its subalgebra $A(1;0)$ with coefficients in Witt or special superalgebras by virtue of the direct sum decomposition of submodules and the weight space decompositions of these submodules relative to their standard Cartan subalgebra. But there are still few general results on cohomology of non-simple classical Lie superalgebras with coefficients in non-trivial modules.

The original motivation for this paper comes from the researches of Chaowen Zhang and his students (see [15, 16]). The treatment of linear superalgebras necessitates matrix computational techniques which are set forth in [9]. The aim of this paper is to determine the first cohomology of $\mathfrak{gl}_{0|2}$ with coefficients in the generalized Witt Lie superalgebra W by applying the methods of matrix representation. This paper contains a considerable amount of computation and its result is easy to understand. Section 2 reviews the necessary notions. Section 3 calculates derivations and inner derivations from $\mathfrak{gl}_{0|2}$ into each irreducible submodule of W and determines the first cohomology of $\mathfrak{gl}_{0|2}$ with coefficients in W over a field of prime characteristic. In Section 4, the first cohomology of $\mathfrak{gl}_{0|2}$ with coefficients in W over a field of characteristic zero is considered.

2. Preliminaries

Throughout this paper $\mathbb{F}$ will be assume to be an arbitrarily field. In addition to the standard notation $\mathbb{Z}$, the set of positive integers and the set of nonnegative integers will be written by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. Let L be a Lie algebra over $\mathbb{F}$ and A be an arbitrary L-module. Denote the $\mathbb{F}$-space of n-linear mappings from the n-fold Cartesian product of L into A by $C^n(L, A)$ $(n > 0)$. We put $C^0(L, A) = A$ and $C^n(L, A) = 0$ for $n < 0$. Then $C^n(L, A)$ is an L-module (see [5]).

For all integers n, an L-module homomorphism $\delta_n : C^n(L, A) \to C^{n+1}(L, A)$ is called the linear coboundary operator if the following formulas hold:

$$\begin{aligned} \delta_n &= 0, && \text{if } n < -1, \\ \delta_{n+1}\delta_n &= 0, && \text{if } n \geq 0, \\ \delta_0(a)(x) &= x \cdot a, && \text{for all } x \in L, a \in A. \end{aligned}$$

The n-th cohomology of L with coefficients in A is defined to be $\mathrm{Ker}\delta_n/\mathrm{Im}\delta_{n-1}$ and denoted by $\mathrm{H}^n(L, A)$ for $n \in \mathbb{N}_0$. The elements of $C^n(L, A)$ are called singular n-cochains with coefficients in A, and the δ_n are referred to as the n-th coboundary operators. Elements of $\mathrm{Ker}\delta_n$ and $\mathrm{Im}\delta_{n-1}$ are called n-cocycles and n-coboundaries, respectively.

An $\mathbb{F}$-linear mapping $\varphi : L \to A$ is called a derivation from L into A if

$$\varphi([x, y]) = x \cdot \varphi(y) - y \cdot \varphi(x)$$

for any $x, y \in L$. Denote by $\mathrm{Der}(L, A)$ the derivation space from L into A. A derivation $\psi_a : L \to A$ is called inner if there is $a \in A$ such that $\psi_a(x) = x \cdot a$ for any $x \in L$. Denote by $\mathrm{Ider}(L, A)$ the inner derivation space from L into A.

It is clear that the derivations from L into A are just the 1-cocycles and the inner derivations from L into A are just 1-coboundaries. Thus

$$\mathrm{H}^1(L, A) \cong \mathrm{Der}(L, A)/\mathrm{Ider}(L, A).$$

The following lemma which is a standard and simple fact will frequently be used in the sequel. For the precise proof please refer to [8].

Lemma 2.1. *Suppose that A is an L-module and $A_1, A_2, \ldots, A_k$ are submodules of A such that $A = A_1 \bigoplus A_2 \bigoplus \cdots \bigoplus A_k$. Then*

$$\mathrm{H}^n(L, A) = \bigoplus_{i=1}^{k} \mathrm{H}^n(L, A_i), \quad n \in \mathbb{N}_0.$$

3. The generalized Witt Lie superalgebra in prime characteristic

Following [17], the notion of the generalized Witt Lie superalgebra W over a field $\mathbb{F}$ of prime characteristic will be recalled. For sake of simplicity, let m, n denote fixed integers in $\mathbb{N} \setminus \{1, 2\}$ without notice, although sometimes a weaker hypothesis is sufficient. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, we put $|\alpha| := \sum_{i=1}^m \alpha_i$. Let $\mathcal{O}(m, \underline{t})$ denote the divided power algebra over $\mathbb{F}$ with a $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{A}(m, \underline{t})\}$, where

$$\mathbb{A}(m, \underline{t}) := \left\{\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \mid 0 \le \alpha_i \le p^{t_i} - 1, i = 1, 2, \ldots, m\right\}.$$

Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ in n variables $\xi_1, \xi_2, \ldots, \xi_n$ and $\mathcal{O}(m, n, \underline{t})$ denote the tenser product $\mathcal{O}(m, \underline{t}) \otimes_{\mathbb{F}} \Lambda(n)$.

For $g \in \mathcal{O}(m, \underline{t})$, $f \in \Lambda(n)$, we write gf for $g \otimes f$. The following formulas hold in $\mathcal{O}(m, n, \underline{t})$:

$$\begin{aligned} x^{(\alpha)} x^{(\beta)} &= \binom{\alpha + \beta}{\alpha} x^{(\alpha+\beta)} \quad \text{for } \alpha, \beta \in \mathbb{N}_0^m; \\ \xi_i \xi_j &= -\xi_j \xi_i \quad \text{for } i, j = 1, 2, \ldots, n; \\ x^{(\alpha)} \xi_j &= \xi_j x^{(\alpha)} \quad \text{for } \alpha \in \mathbb{N}_0^m, j = 1, 2, \ldots, n, \end{aligned}$$

where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^m \binom{\alpha_i+\beta_i}{\alpha_i}$. Put $\mathrm{Y}_0 := \{1, 2, \ldots, m\}$ and $\mathrm{Y}_1 := \{1, 2, \ldots, n\}$. Set

$$\mathbb{B}_k := \{\langle i_1, i_2, \ldots, i_k \rangle \mid 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

and $\mathbb{B} := \bigcup_{k=0}^n \mathbb{B}_k$, where $\mathbb{B}_0 := \emptyset$. For $u = \langle i_1, i_2, \ldots, i_k \rangle \in \mathbb{B}_k$, set $|u| := k$, $|\emptyset| := 0$, $\xi^{\emptyset} := 1$, $\xi^u := \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k}$ and $\xi^E := \xi_1 \xi_2 \cdots \xi_n$. If $u \in \mathbb{B}_k$, $j \in \{u\}$, we suppose that $u - \langle j \rangle \in \mathbb{B}_{k-1}$ such that $\{u - \langle j \rangle\} = \{u\} \setminus \{j\}$. Let $u(j) = |\{l \in \{u\} \mid l < j\}|$. If $j \in \mathrm{Y}_1 \setminus \{u\}$, then we put $u(j) = 0$ and $\xi^{u - \langle j \rangle} = 0$. Clearly, $\left\{x^{(\alpha)} \xi^u \mid \alpha \in \mathbb{A}(m, \underline{t}), u \in \mathbb{B}\right\}$ constitutes an $\mathbb{F}$-basis of $\mathcal{O}(m, n, \underline{t})$.

Let $D_1, \ldots, D_m, d_1, \ldots, d_n$ be linear transformations of $\mathcal{O}(m, n, \underline{t})$ and $\varepsilon_i := (\delta_{i1}, \ldots, \delta_{im})$ such that

$$D_i(x^{(\alpha)}\xi^u) = x^{(\alpha-\varepsilon_i)}\xi^u, \quad i \in \mathrm{Y}_0; \quad d_j(x^{(\alpha)}\xi^u) = (-1)^{u(j)}x^{(\alpha)}\xi^{u-\langle j\rangle}, \quad j \in \mathrm{Y}_1,$$

where δ_{ij} is Kronecker delta, defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Set

$$W := \left\{ \sum_{i=1}^{m} f_i D_i + \sum_{j=1}^{n} f_j d_j \;\middle|\; f_i, f_j \in \mathcal{O}(m, n, \underline{t}), i \in \mathrm{Y}_0, j \in \mathrm{Y}_1 \right\}.$$

Then W is the generalized Witt Lie superalgebra over a field $\mathbb{F}$ of prime characteristic, which is contained in $\mathrm{Der}(\mathcal{O}(m, n, \underline{t}))$. In particular, it is a finite-dimensional simple Lie superalgebra (see [17]). Clearly, W possesses a standard $\mathbb{F}$-basis

$$\left\{ x^{(\alpha)}\xi^u D_i, x^{(\alpha)}\xi^u d_j \mid \alpha \in \mathbb{A}(m, \underline{t}), u \in \mathbb{B}, i \in \mathrm{Y}_0, j \in \mathrm{Y}_1 \right\}.$$

According to the $\mathbb{F}$-basis of W, it is easy to know that

$$W = \mathcal{W} \bigoplus \mathfrak{W},$$

where

$$\begin{aligned}
\mathcal{W} &= \bigoplus_{i=0}^{n} \mathcal{W}_i = \bigoplus_{i=0}^{n} \left\langle x^{(\alpha)}\xi^u D_k \mid \alpha \in \mathbb{A}(m, \underline{t}), u \in \mathbb{B}, |u| = i, k \in \mathrm{Y}_0 \right\rangle, \\
\mathfrak{W} &= \bigoplus_{j=0}^{n} \mathfrak{W}_j = \bigoplus_{j=0}^{n} \left\langle x^{(\alpha)}\xi^u d_l \mid \alpha \in \mathbb{A}(m, \underline{t}), u \in \mathbb{B}, |u| = j, l \in \mathrm{Y}_1 \right\rangle.
\end{aligned}$$

A easy verification shows that $\mathfrak{gl}_{0|2}$ is in fact a Lie algebra which is isomorphic to the subalgebra $\langle \xi_i d_j \mid i, j = 1, 2\rangle$ of W. Then W can be regarded as a $\mathfrak{gl}_{0|2}$-module by means of the adjoint representation. Thus there exists the n-th cohomology of $\mathfrak{gl}_{0|2}$ with coefficients in W.

4. $\mathrm{H}^1(\mathfrak{gl}_{0|2}, W)$ in prime characteristic

In this section, we will deal with $\mathrm{H}^1(\mathfrak{gl}_{0|2}, W)$ in prime characteristic.

Hereafter, let α be an element of $\mathbb{A}(m, \underline{t})$, k be an element of Y_0, u, v be members of set $\mathbb{B}$ and $\{\xi_i d_j \mid i, j = 1, 2\}$ be a standard basis of $\mathfrak{gl}_{0|2}$ without notice. The notation $x^{(\alpha)}$ in the $\mathbb{F}$-basis of W will be omitted since it is not valid under the adjoint operation of $\mathfrak{gl}_{0|2}$. For any $a_i, b_j \in \mathbb{F}$ and $i, j \in \{1, 2, 3, 4\}$, the matrix is written as follow.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ \underbrace{a_4}_{m} & b_4 \end{pmatrix}_{4\times(m+1)} := \begin{pmatrix} a_1 & a_1 & \cdots & a_1 & b_1 \\ a_2 & a_2 & \cdots & a_2 & b_2 \\ a_3 & a_3 & \cdots & a_3 & b_3 \\ a_4 & a_4 & \cdots & a_4 & b_4 \end{pmatrix}_{4\times(m+1)}.$$

It is sufficient to consider $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathcal{W}_i)$ and $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_i)$, where $i = 0, 1, \ldots, n$. On the one hand, we deal with the first cohomology $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathcal{W}_i)$.

Proposition 4.1. $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathcal{W}_0) = 0$

Proof. An easy verification shows that $[x, \mathcal{W}_0] = 0$ for every $x \in \mathfrak{gl}_{0|2}$. If φ is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathcal{W}_0$, then we can easily obtain that $\varphi = 0$, i.e., $\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathcal{W}_0) = 0$. The proof is completed. $\square$

Proposition 4.2. *The following statement holds:*

$$\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathcal{W}_1) = \{\mathcal{A} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_1)\},$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \\ \underbrace{0}_{m} & \underbrace{0}_{m} & \underbrace{c_3}_{m} & \underbrace{c_4}_{m} & \cdots & \underbrace{c_n}_{m} \end{pmatrix}_{4\times nm}, \quad c_3, c_4, \ldots, c_n \in \mathbb{F}.$$

Proof. We know that $\mathcal{W}_1 = \langle \xi_i D_k \rangle$ for $i \in \mathrm{Y}_1$. It suffices to consider $\mathrm{Der}(\mathfrak{gl}_{0|2}, \langle \xi_i D_k \rangle)$ and $\mathrm{Ider}(\mathfrak{gl}_{0|2}, \langle \xi_i D_k \rangle)$. For convenience, we may denote $\xi_i D_1, \xi_i D_2, \ldots, \xi_i D_m$ by $\xi_i D_k$, $i \in \mathrm{Y}_1$, since $[x, \xi_i D_1]$, $[x, \xi_i D_2]$, $\ldots$, $[x, \xi_i D_m]$ are all equal to $[x, \xi_i D_k]$ for any $x \in \mathfrak{gl}_{0|2}$.

(i) The derivation space from $\mathfrak{gl}_{0|2}$ into $\langle \xi_i D_k \mid i \in \mathrm{Y}_1 \rangle$.

Suppose that

$$\varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ c_{31} & c_{32} & \cdots & c_{3n} \\ c_{41} & c_{42} & \cdots & c_{4n} \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix}$$

is a derivation from $\mathfrak{gl}_{0|2}$ into $\langle \xi_i D_k \mid i \in \mathrm{Y}_1 \rangle$, where $c_{ij} \in \mathbb{F}$, $i \in \{1, 2, 3, 4\}$, $j \in \mathrm{Y}_1$. According to the definition of derivations, we have

$$\varphi([x, y]) = [x, \varphi(y)] - [y, \varphi(x)], \quad \forall x, y \in \mathfrak{gl}_{0|2}. \tag{$*$}$$

Substituting x, y by the standard basis of $\mathfrak{gl}_{0|2}$ in Eq. $(*)$, the following equalities hold.

$$\begin{aligned}
\varphi([\xi_1 d_2, \xi_2 d_1]) &= [\xi_1 d_2, \varphi(\xi_2 d_1)] - [\xi_2 d_1, \varphi(\xi_1 d_2)] = \varphi(\xi_1 d_1 - \xi_2 d_2), && (4.1) \\
\varphi([\xi_1 d_2, \xi_1 d_1]) &= [\xi_1 d_2, \varphi(\xi_1 d_1)] - [\xi_1 d_1, \varphi(\xi_1 d_2)] = -\varphi(\xi_1 d_2), && (4.2) \\
\varphi([\xi_1 d_2, \xi_2 d_2]) &= [\xi_1 d_2, \varphi(\xi_2 d_2)] - [\xi_2 d_2, \varphi(\xi_1 d_2)] = \varphi(\xi_1 d_2), && (4.3) \\
\varphi([\xi_2 d_1, \xi_1 d_1]) &= [\xi_2 d_1, \varphi(\xi_1 d_1)] - [\xi_1 d_1, \varphi(\xi_2 d_1)] = \varphi(\xi_2 d_1), && (4.4) \\
\varphi([\xi_2 d_1, \xi_2 d_2]) &= [\xi_2 d_1, \varphi(\xi_2 d_2)] - [\xi_2 d_2, \varphi(\xi_2 d_1)] = -\varphi(\xi_2 d_1), && (4.5) \\
\varphi([\xi_1 d_1, \xi_2 d_2]) &= [\xi_1 d_1, \varphi(\xi_2 d_2)] - [\xi_2 d_2, \varphi(\xi_1 d_1)] = 0. && (4.6)
\end{aligned}$$

An easy computation of Eq. (4.1) shows that

$$\begin{aligned}
& [\xi_1 d_2, c_{21}\xi_1 D_k + c_{22}\xi_2 D_k + \cdots + c_{2n}\xi_n D_k] \\
& -[\xi_2 d_1, c_{11}\xi_1 D_k + c_{12}\xi_2 D_k + \cdots + c_{1n}\xi_n D_k] \\
= \; & (c_{31} - c_{41})\xi_1 D_k + (c_{32} - c_{42})\xi_2 D_k + \cdots + (c_{3n} - c_{4n})\xi_n D_k
\end{aligned}$$

$$= \quad (c_{22})\xi_1 D_k - (c_{11})\xi_2 D_k.$$

Comparing the coefficients of the equality above, we obtain that

$$\begin{cases} c_{22} = c_{31} - c_{41} \\ c_{11} = c_{42} - c_{32} \\ c_{33} = c_{43} \\ \quad\vdots \\ c_{3n} = c_{4n}. \end{cases}$$

Analogously, Eqs. (4.2)–(4.6) show that

$$\begin{cases} c_{32} = c_{12} = \cdots = c_{1n} = 0 \\ c_{21} = c_{23} = \cdots = c_{2n} = 0 \\ c_{41} = c_{32} = 0 \\ c_{31} = c_{22} \\ c_{42} = c_{11}. \end{cases}$$

Therefore,

$$\varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{11} & 0 & 0 & \cdots & 0 \\ 0 & c_{22} & 0 & \cdots & 0 \\ c_{22} & 0 & c_{33} & \cdots & c_{3n} \\ 0 & c_{11} & c_{33} & \cdots & c_{3n} \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix},$$

where $c_{11}, c_{22}, c_{33}, \ldots, c_{3n} \in \mathbb{F}$.

(ii) The inner derivation space from $\mathfrak{gl}_{0|2}$ into $\langle \xi_i D_k \mid i \in \mathrm{Y}_1 \rangle$.

By the definition of inner derivation, elements of $\{\psi_{\xi_i D_k} \mid i \in \mathrm{Y}_1\}$ are as follows:

$$\psi_{\xi_1 D_k} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix},$$

$$\psi_{\xi_2 D_k} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix},$$

$$\psi_{\xi_i D_k} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix}, \quad i \in \mathrm{Y}_1 \setminus \{1, 2\}.$$

Hence

$$\psi_{\xi_i D_k} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ c_2 & 0 & 0 & \cdots & 0 \\ 0 & c_1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix}, \quad c_1, c_2 \in \mathbb{F},\ i \in \mathrm{Y}_1.$$

Combining (i) and (ii), the proof is completed. $\square$

Proposition 4.3. $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathcal{W}_2) = 0$.

Proof. Recall that $\mathcal{W}_2 = \langle \xi_i\xi_j D_k \mid i,j \in \mathrm{Y}_1, i<j\rangle$. In the case of $i,j \in \mathrm{Y}_1 \setminus \{1,2\}$, we have $[x, \mathcal{W}_2] = 0$ for any $x \in \mathfrak{gl}_{0|2}$. Suppose that φ is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathcal{W}_2$, then $\varphi = 0$. For $i \in \{1,2\}$, $\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathcal{W}_2)$ and $\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_2)$ will be considered, respectively.

Suppose that

$$\varphi\begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{12} & \cdots & c_{1n} & b_{13} & \cdots & b_{1n} \\ c_{22} & \cdots & c_{2n} & b_{23} & \cdots & b_{2n} \\ c_{32} & \cdots & c_{3n} & b_{33} & \cdots & b_{3n} \\ c_{42} & \cdots & c_{4n} & b_{43} & \cdots & b_{4n} \end{pmatrix} \begin{pmatrix} \xi_1\xi_2 D_k \\ \vdots \\ \xi_1\xi_n D_k \\ \xi_2\xi_3 D_k \\ \vdots \\ \xi_2\xi_n D_k \end{pmatrix},$$

is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathcal{W}_2$, where $c_{ij}, b_{it} \in \mathbb{F}$, $i \in \{1,2,3,4\}$, $j \in \mathrm{Y}_1 \setminus \{1\}$ and $t \in \mathrm{Y}_1 \setminus \{1,2\}$. It follows from a comparison of the coefficients of Eqs. (4.1)–(4.6) that

$$\begin{cases} c_{12} = c_{43} = \cdots = c_{4n} = 0 \\ c_{22} = \cdots = c_{2n} = 0 \\ b_{33} = \cdots = b_{3n} = 0 \\ b_{13} = \cdots = b_{1n} = 0 \\ c_{32} = c_{42} \\ c_{3t} = b_{2t}, c_{1t} = b_{4t}. \end{cases}$$

Then

$$\begin{aligned} &\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathcal{W}_2) \\ =& \left\{ \varphi \,\middle|\, \varphi\begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & c_{13} & \cdots & c_{1n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & c_{33} & \cdots & c_{3n} \\ c_{32} & c_{33} & \cdots & c_{3n} & 0 & \cdots & 0 \\ c_{32} & 0 & \cdots & 0 & c_{13} & \cdots & c_{1n} \end{pmatrix} \begin{pmatrix} \xi_1\xi_2 D_k \\ \vdots \\ \xi_1\xi_n D_k \\ \xi_2\xi_3 D_k \\ \vdots \\ \xi_2\xi_n D_k \end{pmatrix} \right\}, \end{aligned}$$

where $c_{13}, \ldots, c_{1n}, c_{32}, \ldots, c_{3n} \in \mathbb{F}$.

Using the methods in Proposition 4.2 (ii), the inner derivation space from $\mathfrak{gl}_{0|2}$ into $\mathcal{W}_2$ is as follow.

$$\begin{aligned} &\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_2) \\ =& \left\{ \psi \,\middle|\, \psi\begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & c_{13} & \cdots & c_{1n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & c_{33} & \cdots & c_{3n} \\ c_{32} & c_{33} & \cdots & c_{3n} & 0 & \cdots & 0 \\ c_{32} & 0 & \cdots & 0 & c_{13} & \cdots & c_{1n} \end{pmatrix} \begin{pmatrix} \xi_1\xi_2 D_k \\ \vdots \\ \xi_1\xi_n D_k \\ \xi_2\xi_3 D_k \\ \vdots \\ \xi_2\xi_n D_k \end{pmatrix} \right\}, \end{aligned}$$

where $c_{13}, \ldots, c_{1n}, c_{32}, \ldots, c_{3n} \in \mathbb{F}$. In conclusion, $\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathcal{W}_2) = \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_2)$, that is $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathcal{W}_2) = 0$. $\square$

The following corollary may be proved by similar methods which are used in Proposition 4.3.

Corollary 4.4. $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathcal{W}_i) = 0$, $i \in \mathrm{Y}_1 \setminus \{1\}$.

Theorem 4.5. $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathcal{W}) = \{\mathcal{A} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_1)\}$, *where*

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \\ \underbrace{0}_{m} & \underbrace{0}_{m} & \underbrace{c_3}_{m} & \underbrace{c_4}_{m} & \cdots & \underbrace{c_n}_{m} \end{pmatrix}_{4\times nm}, \quad c_3, c_4, \cdots, c_n \in \mathbb{F}.$$

This is a direct consequence of Lemma 2.1, Propositions 4.1, 4.2, 4.3 and Corollary 4.4. On the other hand, the first cohomology $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_i)$ will be calculated.

Proposition 4.6. *The following statement holds:*

$$\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_0) = \{\mathcal{B} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)\},$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \end{pmatrix}_{4\times n}, \quad c_3, c_4, \cdots, c_n \in \mathbb{F}.$$

Proof. Applying the methods of Proposition 4.2, we will consider $\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)$ and $\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)$, where $\mathfrak{W}_0 = \langle d_l \mid l \in \mathrm{Y}_1 \rangle$.

(i) The derivation space from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_0$.

Suppose that

$$\varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ c_{31} & c_{32} & \cdots & c_{3n} \\ c_{41} & c_{42} & \cdots & c_{4n} \end{pmatrix}_{4\times n} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_0$, where $c_{ij} \in \mathbb{F}$, $i \in \{1,2,3,4\}$, $j \in \mathrm{Y}_1$. By comparing the coefficients of Eqs. (4.1)–(4.6), we can easily show that

$$\begin{cases} c_{32} = c_{11} = c_{13} = \cdots = c_{1n} = 0 \\ c_{41} = c_{22} = c_{23} = \cdots = c_{2n} = 0 \\ c_{12} = c_{31} \\ c_{21} = c_{42} \\ c_{3t} = c_{4t}, \end{cases}$$

where $t \in \mathrm{Y}_1 \setminus \{1,2\}$. Then

$$\begin{aligned} &\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0) \\ = &\left\{ \varphi \,\middle|\, \varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & c_{12} & 0 & \cdots & 0 \\ c_{21} & 0 & 0 & \cdots & 0 \\ c_{12} & 0 & c_{33} & \cdots & c_{3n} \\ 0 & c_{21} & c_{33} & \cdots & c_{3n} \end{pmatrix}_{4\times n} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \right\}, \end{aligned}$$

where $c_{12}, c_{21}, c_{33}, \ldots, c_{3n} \in \mathbb{F}$.

(ii) The inner derivation space from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_0$.

It follows from the methods of Proposition 4.2 (ii) that

$$\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0) = \left\{ \psi \,\middle|\, \psi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 & \cdots & 0 \\ b & 0 & 0 & \cdots & 0 \\ a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \end{pmatrix}_{4\times n} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \right\},$$

where $a, b \in \mathbb{F}$.

Combining (i) and (ii), the desired result follows. □

Proposition 4.7. *The following statement holds:*

$$\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_1) = \{\mathcal{C} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1)\},$$

where

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ a & a & c_3 & \cdots & c_n & 0 \\ a & a & \underbrace{c_3}_{n-2} & \cdots & \underbrace{c_n}_{n-2} & \underbrace{0}_{4(n-2)+2} \end{pmatrix}_{4\times n^2}, \quad a, c_3, \ldots, c_n \in \mathbb{F}.$$

Proof. It is easy to show that

$$\mathcal{C} \cong \mathcal{C}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a & a & 0 & c_3 & \cdots & c_n \\ 0 & 0 & a & a & \underbrace{0}_{4(n-2)} & \underbrace{c_3}_{n-2} & \cdots & \underbrace{c_n}_{n-2} \end{pmatrix}_{4\times n^2},$$

for $a, c_3, \ldots, c_n \in \mathbb{F}$. If $\mathfrak{W}_1 = \langle \xi_i d_l \mid i \in \mathrm{Y}_1, l \in \mathrm{Y}_1 \setminus \{1,2\}\rangle$, then the $4 \times (n-2)^2$ matrix in the right part of $\mathcal{C}'$ may be easily obtained by Proposition 4.2. It is enough to consider the case of $l \in \{1, 2\}$. Then $\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1')$ and $\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1')$ will be calculated by the methods of Proposition 4.2, where $\mathfrak{W}_1' := \langle \xi_i d_l \mid i \in \mathrm{Y}_1, l \in \{1, 2\}\rangle$.

(i) The derivation space from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_1'$.

Let $c_{ij} \in \mathbb{F}$, $i \in \{1, 2, 3, 4\}$, $j \in \{1, 2, \ldots, 6\}$ and $t \in \mathrm{Y}_1 \setminus \{1, 2\}$. Suppose that

$$\varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \end{pmatrix} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \\ \xi_t d_1 \\ \xi_t d_2 \end{pmatrix},$$

is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_1'$.

Coefficients of Eqs. (4.1)–(4.6) show that

$$\begin{cases} c_{12} = c_{15} = c_{21} = c_{26} = c_{36} = c_{45} = 0 \\ c_{33} = c_{34} = c_{43} = c_{44} \\ c_{13} = -c_{14} = -c_{32} = c_{42} \\ c_{23} = -c_{24} = -c_{31} = c_{41} \\ c_{11} = -c_{22} \\ c_{16} = c_{35} \\ c_{25} = c_{46}. \end{cases}$$

Hence

$$\begin{aligned} &\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1') \\ = &\left\{ \varphi \,\middle|\, \varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{11} & 0 & c_{13} & -c_{13} & 0 & c_{16} \\ 0 & -c_{11} & c_{23} & -c_{23} & c_{25} & 0 \\ -c_{23} & -c_{13} & c_{33} & c_{33} & c_{16} & 0 \\ c_{23} & c_{13} & c_{33} & c_{33} & 0 & c_{25} \end{pmatrix} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \\ \xi_t d_1 \\ \xi_t d_2 \end{pmatrix} \right\}, \end{aligned}$$

where $c_{ij} \in \mathbb{F}$, $i \in \{1, 2, 3\}$, $j \in \mathrm{Y}_1$ and $t \in \mathrm{Y}_1 \setminus \{1, 2\}$.

(ii) The inner derivation space from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_1'$.

Using the methods in Proposition 4.2 (ii), the inner derivation space can be represented by matrices as follow.

$$\begin{aligned} &\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1') \\ = &\left\{ \psi \,\middle|\, \psi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} a & 0 & d & -d & 0 & m \\ 0 & -a & e & -e & n & 0 \\ -e & -d & 0 & 0 & m & 0 \\ e & d & 0 & 0 & 0 & n \end{pmatrix} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \\ \xi_t d_1 \\ \xi_t d_2 \end{pmatrix} \right\}, \end{aligned}$$

where $a, b, c, d, e, m, n \in \mathbb{F}$ and $t \in \mathrm{Y}_1 \setminus \{1, 2\}$.

Proposition 4.2 and results above imply that

$$\begin{aligned} \mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_1) &= \mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1)/\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1) \\ &= \{\mathcal{C}' + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1)\}. \end{aligned}$$

This completes the proof. □

Proposition 4.8. *The following statement holds:*

$$\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_2) = \{\mathcal{D}_2 + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_2)\},$$

where

$$\mathcal{D}_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ c & c & 0 & \cdots & 0 \\ \underbrace{c}_{C_{n-2}^1} & \underbrace{c}_{C_{n-2}^1} & 0 & \cdots & 0 \end{pmatrix}_{4 \times nC_n^2}, \quad c \in \mathbb{F}.$$

Proof. In this proof, we suppose that $t \in \mathrm{Y}_1 \setminus \{1,2\}$, $|v| = 2$ and $\xi_1, \xi_2 \notin \xi^v$. It suffices to consider derivations and inner derivations from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_2' := \langle \xi^u d_l \mid |u| = 2, l \in \{1,2\}\rangle$. By similar methods used in Proposition 4.2, the computation of $\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_2')$ and $\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_2')$ is routine. Then

$$\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_2') = \left\{ \varphi \;\middle|\; \varphi\begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \mathfrak{A}\begin{pmatrix} \xi_1\xi_2 d_1 \\ \xi_1\xi_2 d_2 \\ \xi_1\xi_t d_1 \\ \xi_1\xi_t d_2 \\ \xi_2\xi_t d_1 \\ \xi_2\xi_t d_2 \\ \xi^v d_1 \\ \xi^v d_2 \end{pmatrix} \right\},$$

$$\begin{aligned} &\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_2') \\ = &\left\{ \varphi \;\middle|\; \varphi\begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & a & d & f & 0 & -d & 0 & g \\ b & 0 & e & 0 & -f & -e & h & 0 \\ 0 & -b & 0 & -e & -d & 0 & g & 0 \\ -a & 0 & 0 & e & d & 0 & 0 & h \end{pmatrix}\begin{pmatrix} \xi_1\xi_2 d_1 \\ \xi_1\xi_2 d_2 \\ \xi_1\xi_t d_1 \\ \xi_1\xi_t d_2 \\ \xi_2\xi_t d_1 \\ \xi_2\xi_t d_2 \\ \xi^v d_1 \\ \xi^v d_2 \end{pmatrix} \right\}, \end{aligned}$$

where

$$\mathfrak{A} = \begin{pmatrix} 0 & c_{12} & c_{13} & c_{14} & 0 & -c_{13} & 0 & c_{18} \\ c_{21} & 0 & c_{23} & 0 & -c_{14} & -c_{23} & c_{27} & 0 \\ 0 & -c_{21} & c_{33} & -c_{23} & -c_{13} & c_{33} & c_{18} & 0 \\ -c_{12} & 0 & c_{33} & c_{23} & c_{13} & c_{33} & 0 & c_{27} \end{pmatrix},$$

$a, b, d, e, f, g, h, c_{ij} \in \mathbb{F}$, $i \in \{1,2,3\}$ and $j \in \{1,2,\ldots,8\}$.

Combining with Proposition 4.3, we have

$$\begin{aligned} \mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_2) &= \mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_2)/\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_2) \\ &= \{\mathcal{D} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_2)\}, \end{aligned}$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c & 0 & 0 & c & 0 & \cdots & 0 \\ 0 & 0 & \underbrace{c}_{C_{n-2}^1} & \underbrace{0}_{C_{n-2}^1} & \underbrace{0}_{C_{n-2}^1} & \underbrace{c}_{C_{n-2}^1} & 0 & \cdots & 0 \end{pmatrix}_{4 \times nC_n^2}, \quad c \in \mathbb{F}.$$

It follows from changing the order of the basis of $\mathfrak{W}_2$ that $\mathcal{D} \cong \mathcal{D}_2$. The proof is completed. □

The following corollary will be used to represent $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_i)$. In fact, it may be easily proved by induction on i.

Corollary 4.9. *For* $i \in \mathrm{Y}_1 \setminus \{1, n\}$ *and* $c \in \mathbb{F}$,

$$\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_i) = \{\mathcal{D}_i + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_i)\},$$

where

$$\mathcal{D}_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ c & c & 0 & \cdots & 0 \\ \underbrace{c}_{C_{n-2}^{i-1}} & \underbrace{c}_{C_{n-2}^{i-1}} & 0 & \cdots & 0 \end{pmatrix}_{4\times nC_n^i}.$$

Proposition 4.10. $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_n) = 0$.

Proof. The methods employed in Proposition 4.3 may verify that

$$\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_n) \cong \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_n).$$

This implies that $\mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}_n)$ vanishes. □

Theorem 4.11. *The following statement holds:*

$$\begin{aligned} \mathrm{H}^1(\mathfrak{gl}_{0|2}, \mathfrak{W}) \quad = \quad & \{\mathcal{B} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)\} \bigoplus \{\mathcal{C} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1)\} \\ & \bigoplus \bigoplus_{i=2}^{n-1} \{\mathcal{D}_i + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_i)\}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{B} \quad &= \quad \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \end{pmatrix}_{4\times n}, \\ \mathcal{C} \quad &= \quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ a & a & c_3 & \cdots & c_n & 0 \\ a & a & \underbrace{c_3}_{n-2} & \cdots & \underbrace{c_n}_{n-2} & \underbrace{0}_{4(n-2)+2} \end{pmatrix}_{4\times n^2}, \\ \mathcal{D}_i \quad &= \quad \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ c & c & 0 & \cdots & 0 \\ \underbrace{c}_{C_{n-2}^{i-1}} & \underbrace{c}_{C_{n-2}^{i-1}} & 0 & \cdots & 0 \end{pmatrix}_{4\times nC_n^i}, \quad a, c, c_3, c_4, \ldots, c_n \in \mathbb{F}. \end{aligned}$$

This is a direct consequence of Lemma 2.1, Propositions 4.6, 4.7, 4.8 and Corollary 4.9.

Theorem 4.12. *The following statement holds:*

$$\begin{aligned} \mathrm{H}^1(\mathfrak{gl}_{0|2}, W) \quad = \quad & \{\mathcal{A} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_1)\} \bigoplus \{\mathcal{B} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)\} \\ & \bigoplus \{\mathcal{C} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1)\} \bigoplus \bigoplus_{i=2}^{n-1} \{\mathcal{D}_i + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_i)\}, \end{aligned}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \\ \underbrace{0}_{m} & \underbrace{0}_{m} & \underbrace{c_3}_{m} & \underbrace{c_4}_{m} & \cdots & \underbrace{c_n}_{m} \end{pmatrix}_{4\times nm},$$

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \end{pmatrix}_{4\times n},$$

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ a & a & c_3 & \cdots & c_n & 0 \\ a & a & \underbrace{c_3}_{n-2} & \cdots & \underbrace{c_n}_{n-2} & \underbrace{0}_{4(n-2)+2} \end{pmatrix}_{4\times n^2},$$

$$\mathcal{D}_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ c & c & 0 & \cdots & 0 \\ \underbrace{c}_{C_{n-2}^{i-1}} & \underbrace{c}_{C_{n-2}^{i-1}} & 0 & \cdots & 0 \end{pmatrix}_{4\times nC_n^i},$$

$a, c, c_3, c_4, \ldots, c_n \in \mathbb{F}$.

This is a direct consequence of Lemma 2.1, Theorems 4.5 and 4.11.

5. $\mathrm{H}^1(\mathfrak{gl}_{0|2}, W)$ in characteristic zero

Imitating the situation that the characteristic number of basic field $\mathbb{F}$ is zero, we consider $\mathrm{H}^1(\mathfrak{gl}_{0|2}, W)$ in this section.

In the below the notion of the generalized Witt Lie superalgebra over a field $\mathbb{F}$ of characteristic zero will be reviewed (see [6, 10]). Let $\mathbb{F}[x_1, \ldots, x_m]$ be the polynomial algebra over $\mathbb{F}$ in m variables $x_1, \ldots, x_m$ and $\Lambda(n)$ as in the case of prime characteristic. Denote the tenser product $\mathbb{F}[x_1, \ldots, x_m] \otimes_{\mathbb{F}} \Lambda(n)$ by $\Lambda(m, n)$. Obviously, $\Lambda(m, n)$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathbb{F}[x_1, \ldots, x_m]$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. For $f \in \mathbb{F}[x_1, \ldots, x_m]$, $g \in \Lambda(n)$, we write fg for $f \otimes g$.

Let $W = \mathrm{Der}(\Lambda(m, n))$. Then W is the generalized Witt Lie superalgebra in characteristic zero. We know that W possesses a standard $\mathbb{F}$-basis

$$\{f_i \xi^u D_i, f_j \xi^u d_j \mid f_i, f_j \in \mathbb{F}[x_1, \ldots, x_m], u \in \mathbb{B}, i \in \mathrm{Y}_0, j \in \mathrm{Y}_1\}.$$

Obviously, W is also a $\mathfrak{gl}_{0|2}$-module by means of the adjoint representation in this situation. Thus there exists the n-th cohomology of $\mathfrak{gl}_{0|2}$ with coefficients in W as well.

The elements of $\mathbb{F}[x_1, \ldots, x_m]$ are not valid under the adjoint operation of $\mathfrak{gl}_{0|2}$. This shows that $\mathrm{H}^1(\mathfrak{gl}_{0|2}, W)$ in characteristic zero can be obtained by the same methods in prime characteristic. Since $\mathrm{H}^1(\mathfrak{gl}_{0|2}, W)$ in prime characteristic have been given by Theorem 4.12, the following theorem can be easily obtained.

Theorem 5.1. *Let* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}_i$, $\mathcal{W}_1$, $\mathfrak{W}_0$, $\mathfrak{W}_1$, $\mathfrak{W}_i$ *as in Theorem 4.12 and* $\mathbb{F}$ *be the underlying base field of characteristic zero. Then we have the following statement:*

$$\mathrm{H}^1(\mathfrak{gl}_{0|2}, W) = \{\mathcal{A} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_1)\} \bigoplus \{\mathcal{B} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)\}$$
$$\bigoplus \{\mathcal{C} + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1)\} \bigoplus \bigoplus_{i=2}^{n-1} \{\mathcal{D}_i + \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_i)\}.$$

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